Equidistribution

A sequence (x_n) on $[0,1)$ is equidistributed if for any interval (a,b),

$$\lim_{N\to\infty} \frac{1}{N} \#\{1\le n\le N : x_n\in(a,b)\} = b-a$$

$\langle x\rangle = x - \lfloor x\rfloor$ denotes the fractional part of x.

Question: Whether $\langle x_n\rangle$ equidistributed or not?

Known results

Equidistributed sequences:

- $\langle n^k\alpha\rangle$, $\alpha\notin\mathbb{Q}$, $k\in\mathbb{N}$
- $\langle n^\sigma\alpha\rangle$, $\sigma>0$, $\sigma\notin\mathbb{N}$
- $\langle\alpha^n\rangle$ for 'many'/typical $\alpha>1$

...

Non-equidistributed sequences:

- $\langle\alpha\log n\rangle$, $\alpha>0$
- $\langle\alpha^n\rangle$ for α satisfying some special algebraic condition
  e.g. Pisot

...

Open Problems:

- $\langle e^n \rangle$
- $\langle \pi^n \rangle$
- $\langle (\frac{3}{2})^n \rangle$

...

Weyl's Criterion

The following are equivalent:

(i) (x_n) is equidistributed

(ii) For $k \in \mathbb{Z} \setminus \{0\}$,

$$\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k x_n} \to 0 \quad \text{as} \quad N \to \infty$$

(iii) For continuous function f,

$$\frac{1}{N} \sum_{n=1}^{N} f(x_n) \to \int_0^1 f(y)\, dy$$

I. $\langle n^\sigma \alpha \rangle$ is equidistributed, $0<\sigma<1$, $\alpha>0$.

Pf: By Weyl's Criterion, it suffices to show for $k \in \mathbb{Z}\setminus\{0\}$

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k\alpha n^\sigma} \to 0 \quad \text{as } N\to\infty.$$

Let $k\alpha = b$.

Step 1: $\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i b n^\sigma} - \frac{1}{N}\int_1^{N+1} e^{2\pi i b x^\sigma}dx\right| \to 0$ as $N\to\infty$

Pf: $\left|\sum_{n=1}^{N} e^{2\pi i b n^\sigma} - \int_1^{N+1} e^{2\pi i b x^\sigma}dx\right|$

$$= \left|\sum_{n=1}^{N} e^{2\pi i b n^\sigma} - \sum_{n=1}^{N}\int_n^{n+1} e^{2\pi i b x^\sigma}dx\right|$$

$$= \left|\sum_{n=1}^{N}\int_n^{n+1}\left(e^{2\pi i b n^\sigma} - e^{2\pi i b x^\sigma}\right)dx\right|$$

$$\le \sum_{n=1}^{N}\int_n^{n+1}\left|e^{2\pi i b n^\sigma} - e^{2\pi i b x^\sigma}\right|dx$$

$$= \sum_{n=1}^{N}\int_n^{n+1}\left|e^{2\pi i b(x^\sigma - n^\sigma)} - 1\right|dx$$

$$= \sum_{n=1}^{N}\int_n^{n+1}\left|e^{\pi i b(x^\sigma - n^\sigma)} - e^{-\pi i b(x^\sigma - n^\sigma)}\right|dx$$

$$= \sum_{n=1}^{N} \int_n^{n+1} \left| 2 \sin \pi b (x^\sigma - n^\sigma) \right| dx$$

$$\le 2\pi b \sum_{n=1}^{N} \int_n^{n+1} |x^\sigma - n^\sigma| \, dx \qquad |\sin x| < |x|$$

$$\le 2\pi b \sum_{n=1}^{N} \left[(n+1)^\sigma - n^\sigma \right]$$

$$= 2\pi b \left[(N+1)^\sigma - 1 \right]$$

$$= o(N) \qquad \sigma < 1$$

Step 2: $\frac{1}{N} \int_1^{N+1} e^{2\pi i b x^\sigma} dx \to 0$ as $N \to \infty$

Pf: $\int_1^{N+1} e^{2\pi i b x^\sigma} dx$

$$= \frac{1}{\sigma} \int_1^{(N+1)^\sigma} e^{2\pi i b y} y^{\frac{1}{\sigma}-1} dx \qquad \begin{aligned} y &= x^\sigma \\ x &= y^{\frac{1}{\sigma}} \\ dx &= \frac{1}{\sigma} y^{\frac{1}{\sigma}-1} dy \end{aligned}$$

$$= \frac{1}{\sigma} \Bigg[\underbrace{\frac{1}{2\pi i b} e^{2\pi i b y} y^{\frac{1}{\sigma}-1} \Bigg|_{y=1}^{(N+1)^\sigma}}_{(\mathrm{I})} - \underbrace{\frac{1}{2\pi i b} \left(\frac{1}{\sigma} - 1 \right) \int_1^{(N+1)^\sigma} e^{2\pi i b y} y^{\frac{1}{\sigma}-2} dy}_{(\mathrm{II})} \Bigg]$$

$$|\mathrm{I}| \lesssim y^{\frac{1}{\sigma}-1}\Big|_{y=1}^{(N+1)^{\sigma}}$$

$$= (N+1)^{1-\sigma} - 1$$

$$= o(N) \qquad \sigma > 0$$

$$|\mathrm{II}| \lesssim \int_1^{(N+1)^{\sigma}} y^{\frac{1}{\sigma}-2}\, dy$$

$$\lesssim y^{\frac{1}{\sigma}-1}\Big|_{y=1}^{(N+1)^{\sigma}}$$

$$= (N+1)^{1-\sigma} - 1$$

$$= o(N)$$

$\square$

II. $\langle \alpha \log n \rangle$ is not equidistributed for any α.

Proof: It suffices to show

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i b \log n} \not\to 0 \quad \text{as} \quad N \to \infty$$

where $b = k\alpha$

Step 1: $\left| \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i b \log n} - \frac{1}{N}\int_1^{N+1} e^{2\pi i \log x}\, dx \right| \to 0$

Pf: $\left| \sum_{n=1}^{N} e^{2\pi i b \log n} - \int_1^{N+1} e^{2\pi i \log x}\, dx \right.$

$$\leq \sum_{n=1}^{N} \int_n^{n+1} \left| e^{2\pi i b \log x - \log n} - 1 \right| dx$$

$$\leq 2\pi b \sum_{n=1}^{N} \int_n^{n+1} \left| \sin(\log x - \log n) \right| dx$$

$$\leq 2\pi b \sum_{n=1}^{N} \log(n+1) - \log n$$

$$= 2\pi b \log(N+1)$$

$$= o(N)$$

Step 2: $\left|\frac{1}{N}\int_1^{N+1} e^{2\pi i \log x}\,dx\right| \not\to 0$

Pf: $\left|\int_1^{N+1} e^{2\pi i \log x}\,dx\right|$

$= \left|\int_0^{\log(N+1)} e^{2\pi i y} e^{y}\,dy\right|$

$y = \log x$
$x = e^y$
$dx = e^y dy$

$= \left|\int_0^{\log(N+1)} e^{(2\pi i+1)y}\,dy\right|$

$= \left|\frac{1}{2\pi i+1} e^{(2\pi i+1)y}\Big|_0^{\log(N+1)}\right|$

$= \left|\frac{1}{2\pi i+1}\left(e^{(2\pi i+1)\log(N+1)} - 1\right)\right|$

$\geq \frac{1}{\sqrt{4\pi^2+1}}\left|e^{(2\pi i+1)\log(N+1)}\right| - 1$

$= \frac{1}{\sqrt{4\pi^2+1}}(N+1-1)$

$= \frac{N}{\sqrt{4\pi^2+1}}$

$\square$